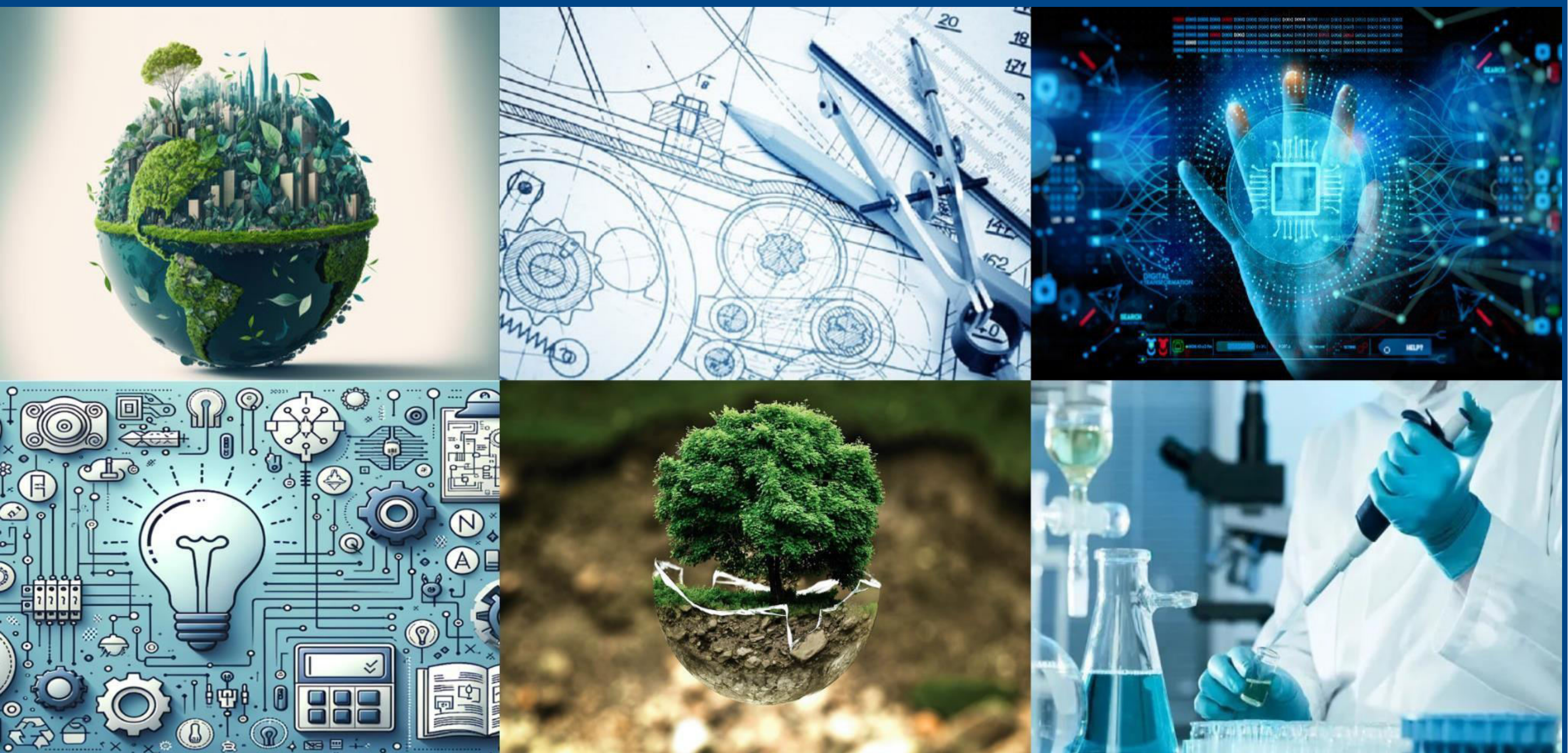
ISSN: 2582-7219

International Journal of Multidisciplinary Research in Science, Engineering and Technology

(A Monthly, Peer Reviewed, Refereed, Scholarly Indexed, Open Access Journal)



Impact Factor: 8.206

Volume 9, Issue 1, January 2026

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ISSN: 2582-7219 | www.ijmrset.com | Impact Factor: 8.206| ESTD Year: 2018|

International Journal of Multidisciplinary Research in Science, Engineering and Technology (IJMRSET)

(A Monthly, Peer Reviewed, Refereed, Scholarly Indexed, Open Access Journal)

Applications of Fixed Point Theorems in Metric Spaces with Binary Relations to Differential Equations

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ABSTRACT: This paper investigates the applications of fixed point theorems in metric spaces equipped with binary relations to ordinary and functional differential equations. By incorporating relational structures into the metric framework, the proposed results weaken traditional contractive conditions and allow greater flexibility in modeling nonlinear phenomena. Several existence and uniqueness results for solutions of differential equations are established under relational contractive mappings. Illustrative examples are provided to demonstrate the effectiveness of the developed theoretical framework and to highlight its applicability to various classes of differential equations. Fixed point theory a fundamental role in nonlinear analysis and provides powerful tools for studying the existence and uniqueness of solutions to differential equations. In recent years, fixed point theorems in metric spaces endowed with binary relations have attracted significant attention due to their ability to unify and generalize classical contraction principles.

KEYWORDS: Fixed point theorem; Metric space; Binary relation; Differential equations; Existence and uniqueness; Nonlinear analysis.

I. INTRODUCTION

Fixed point theory has emerged as one of the most powerful and versatile tools in nonlinear analysis, owing to its wide range of applications across mathematics and applied sciences. At its core, a fixed point theorem guarantees the existence (and sometimes uniqueness) of a point that remains invariant under a given mapping. Classical results such as the Banach contraction principle, Brouwer fixed point theorem, and Schauder fixed point theorem have played a foundational role in the study of functional equations, optimization problems, and, most notably, differential and integral equations.

Metric spaces provide a natural and flexible framework for fixed point theory. The structure induced by a metric allows the precise measurement of distances between elements, enabling the formulation of contraction-type conditions and convergence arguments. Over time, researchers have generalized classical fixed point results by relaxing metric conditions or by enriching the underlying space with additional structures. One such significant generalization involves metric spaces endowed with binary relations. These relations capture order, comparability, or dependency between elements and allow fixed point results to be developed in settings where traditional assumptions, such as continuity or monotonicity, may fail.

The incorporation of binary relations into metric fixed point theory has opened new avenues for applications, particularly in the study of differential equations. Many real-world phenomena modeled by differential equations exhibit inherent relational structures, such as causality, partial ordering, or directional dependence. For example, boundary value problems, functional differential equations, and systems with constraints often involve solutions that satisfy certain relational conditions rather than purely metric ones. Fixed point theorems formulated in metric spaces with binary relations are well-suited to address such problems, as they allow mappings to be contractive or monotone only along related elements instead of the entire space.

In recent years, several authors have extended classical contraction principles to relational metric spaces by introducing concepts such as relation-preserving mappings, weak contractions, and compatibility conditions between the metric and

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the binary relation. These generalized fixed point results have proven effective in establishing existence and uniqueness of solutions for various classes of ordinary and partial differential equations. Moreover, they facilitate the treatment of nonlinear problems where standard Lipschitz conditions are too restrictive or difficult to verify.

The application of fixed point theorems in metric spaces with binary relations to differential equations typically involves transforming the differential problem into an equivalent integral equation or operator equation. The associated operator is then shown to satisfy the hypotheses of an appropriate fixed point theorem under the given relational and metric framework. This approach not only simplifies the analysis but also provides constructive methods for approximating solutions.

The purpose of this work is to explore the applications of fixed point theorems in metric spaces equipped with binary relations to differential equations. Emphasis is placed on highlighting how relational structures enhance classical fixed point techniques and broaden their applicability to complex differential models. Such developments underscore the growing importance of relational fixed point theory as a unifying and powerful tool in modern mathematical analysis.

II. APPLICATIONS OF FIXED POINTS TO THE SOLUTION OF LINEAR AND NONLINEAR INTEGRAL EQUATIONS

A fixed point, as the name implies, is a point on a curve that remains constant regardless of the transformations applied to it. The history of fixed-point theory may be traced back to the development of topology, a significant branch of mathematics, by Johann Benedict Listing towards the close of the nineteenth century. However, at that time, it was just a concept and not a fully formed field of study. During the early 20th century, the concept of topological spaces was introduced. Afterwards, the fixed-point theory was established by French mathematician H. Poincare. Unlike other areas of mathematics, it was not found until much later, although it is now completely mature. Numerous branches of mathematics are included in the study of fixed point life. Classical and functional analysis are both necessary in analysis, general and algebraic topology are both necessary in topology, and knowledge of operator theory may be used to the study of fixed point existence.

A number of mathematical disciplines have relied on fixed points and fixed point theorems to provide a theoretical framework for their work. These include mathematical economics, approximation theory, game theory, mathematical boundary value problems, initial value problems, and theoretical explanations in many others. Throughout mathematics' history, fixed-point theory has consistently shown its significance. Fixed points have seen a meteoric rise in use thanks to new methods that allow for their discovery with greater efficiency and accuracy. It is now utilized almost everywhere and is no longer restricted to the aforementioned areas of mathematics. Linear, differential, integral, and non-linear integral equations, as well as their fixed points, may be solved and located with the help of fixed point theory. As per the fixed-point theory, it may be said that each maps $T: Y \rightarrow Y$ guarantees either one fixed point or more than one fixed point, and that this map is a self-mapping on Y of a topological space. We may further define such locations as y in Y, where $y=Ty$. Now that we have established what a fixed point is, let's look at a few real-world instances.

- If an is not equal to zero, then we can translate any mapping $T(y)= y+a$. Because the criteria for a map to have a fixed point are that for any y, $T(y) \neq y$, for $a \neq 0$, this mapping will never have a fixed point.
- Rotation—When a plane is rotated, there is always going to be one fixed point—the place where the plane's center of rotation is located.
- There will always be just two fixed points in the mapping $y \rightarrow y2$ specified on $R \rightarrow R$. This curve has a fixed point of zero because, at y=0, y 2 = 0, or $T(y) = y$. The fixed point of this curve is similarly 1, since for y = 1, y 2 = 1. That is, $T(y) = y$. This curve will only have two fixed points, namely 0 and 1, since there is no other point for which $T(y) = y$.
- A mapping from R to R, denoted as $y \rightarrow y3$, will also always include only three fixed points. Since $T(y)= y$, the fixed point of this curve is 0, since y=0 y 3 = 0. Additionally, the fixed point of this curve is 1 because, at y=1, y3=1, and $T(y)= y$. Additionally, when y=-1, y3=-1, and $T(y)= y$, we can see that -1 is the fixed point of the curve as well. This curve will only have three fixed points, 0, 1, and -1, since there is no other point for which $T(y) = y$. Additionally, for any value of y, T(y) will always be equal to y for mapping $y \rightarrow y$ defined on $R \rightarrow R$. the number of fixed points in this kind of mapping is consequently limitless.

So far, we have only been successful in locating a maximum of three finite fixed locations.

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Dutch mathematician L.E.J. Brouwer initially proposed the fixed point theorem in 1912. A theorem grounded in algebraic topology underpinned it. Brouwer demonstrated in his theorem of fixed points that any continuous self-map, T, of any closed ball, Rn, with one unit dimension, would inevitably have a fixed point, y, in Y, such that T(y) = y. Following the validation of Brouwer's theorem, two additional seminal theorems, the "Banach fixed point theorems" and the "Banach contraction principle," were established. A unique fixed point exists in entire metric space if and only if the self-mapping is also a contraction mapping, as shown in this theorem. Therefore, the existence and uniqueness of a fixed point may be simply determined using the Banach contraction principle. In the years after the discovery of the Banach contraction principle, several other mathematicians continued to refine and expand upon it. As a consequence, they were able to solve a wide variety of integral equations, including linear, differential, non-linear, and linear systems. Many new applications of fixed-point theory have their roots in the Banach contraction principle.

III. APPLICATION OF FIXED POINT TO REAL-WORLD PROBLEMS

Over the last hundred years or more, fixed-point theory has been an indispensable tool for studying non-linear processes. In modern times, fixed-point theory has found applications in almost every scientific discipline, including but not limited to: astronomy, computer science, biology, engineering, chemistry, and game theory. Different branches of fixed-point theory have made technical strides, and the discipline has many applications and uses across a wide range of real-world challenges. In order to understand generic topology and non-linear functional analysis, fixed-point theory for metric spaces is essential. Conversely, communication engineers also make use of fixed point theory when addressing issues. Some more real-world applications of fixed-point theory include genetics, algorithm testing, and the solving of chemical equations.

Relevance to finding solutions of linear equations: As is well known, there are several direct methods for solving linear equations. More accurate results may also be obtained by using iterative approaches for both solutions. By using the Banach contraction principle, we may determine the fixed point as the solution to a system of linear equations. Let's have a look at how fixed point is used to solve linear equations. Now, consider a set of three linear equations, for example

$a1x1 + b1x2 + c1x3 = d1$

$a2x1 + b2x2 + c2x3 = d2$

$a3x1 + b3x2 + c3x3 = d3$

Additionally, let T be a self-mapping function that is continuous, meaning that T:X→X and d (Tx, Ty) < k d (x, y) are both true. We can now determine the values of $b1$, $b2$, and $b3$ using the aforementioned formulae. So,

$x1 = (1 - a1)\, x1 - b1x2 - c1x3 + d1$

$x2 = (1 - b2\,)2 - a2x1 - c2x3 + d2$

$x3 = (1 - c3\,)3 - b3x2 - a3x1 + d3$

The above set of equations may be simplified to by applying the knocker delta:

$xi = \sum \alpha ijxj + dj$

The provided system of linear equations is shown above as x = Ax + d. The fact that T(x) = Ax + d is continuous proves that all linear mappings are continuous.

d (Tx, Ty) = |Tx – Ty| = |Ax – Ay| = |A| |x-y| = |A| d (x, y)

T is a contraction if and only if |A| is less than 1. In this case, we have used a system of three linear equations as an example, but the result stands regardless of the number of equations in the system. A unique fixed point for $|\beta bb| \leq 1$ will exist in every system of n linear equations, according to the Banach contraction principle.

IV. APPLICATION OF BANACH CONTRACTION THEOREM TO DIFFERENTIAL EQUATIONS

In order to demonstrate that the following ordinary differential equation, given an initial condition, exists and is unique, we use the Banach contraction theorem:

$$\frac{dy}{dx} = f(x, y)\ ,\ y(x_0) = y_0$$

Proposition 7, also known as Picard's theorem: Given a rectangle and two variables x and y, f(x, y) is a continuous function. A = {(x, y) : a ≤ x ≤ b, c ≤ y ≤ d} in the second y-variable meet the Lipschitz criterion.

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Additionally, consider any inside point of A as (x0, y0). Following that, the dynamical system

$$\frac{dy}{dx} = f(x, y)$$

has a singular solution, denoted as y = g(x), that traverses (x0, y0).
Assuming the differential equation is correct, the proof is

$$\frac{dy}{dx} = f(x, y)$$

Assume that y= g(x) and that g(x0) = y0 both hold according to. From x0 to x, we get by integrating.

$$[y]_{x_0}^{x} = \int_{x_0}^{x} f\left(t, g(t)\right) dt$$

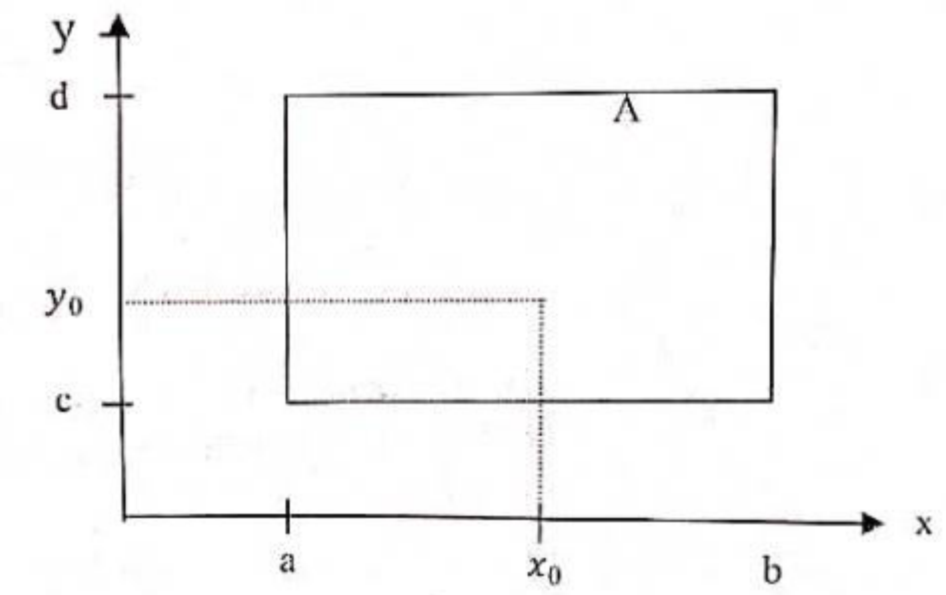
Thus

$$g(x) - g(x_0) = \int_{x_0}^{x} f\left(t, g(t)\right) dt \qquad \because y = g(x).$$

Therefore,

$$g(x) = y_0 + \int_{x_0}^{x} f\left(t, g(t)\right) dt.$$

There is a direct correspondence between a unique solution to and a unique solution to. There is a constant q > 0 such that f(x, y) fulfills the Lipschitz condition in y |f(x, y1) − f(x, y2)| ≤ q|y1 − y2| where (x, y1),(x, y2) ∈ A.



The function f(x, y) is limited because it is continuous on a compact subset A of R^2. Thus, a positive constant m is present such that |f(x, y)| ≤ m, ∀ (x, y) ∈ A.
Let us choose a positive constant p such that pq < 1 and the rectangle
B = {(x, y)|x0 − p ≤ x ≤ x0 + p, y0 − pm ≤ y ≤ y0 + pm}

is not part of A. Enumerate all continuous functions y = g(x) defined on real-valued sets X. [x0 − p, x0 + p] such that ‖g(x) − y0‖ ≤ mp i.e, X is a closed subset of the Banach space C[x0 − p, x0 + p] with the sup norm. Let T : X → X be defined as T g = h where

$$h(x) = y_0 + \int_{x_0}^{x} f\left(t, g(t)\right) dt.$$

Here

$$\begin{aligned} \|h(x) - y_0\| &= \left\| \int_{x_0}^{x} f\left(t, g(t)\right) dt \right\| \\ &\leq \int_{x_0}^{x} \left| f\left(t, g(t)\right) \right| dt \\ &\leq m \int_{x_0}^{x} dt \\ &\leq m(x - x_0) \leq mp \end{aligned}$$

Thus, T is well-defined because h(x) is a member of X.
Let g, g1 ∈ X, then
‖T g − T g1‖ = ‖h − h1‖

$$= \left\| y_0 + \int_{x_0}^{x} f\left(t, g(t)\right) dt - y_0 - \int_{x_0}^{x} f\left(t, g_1(t)\right) dt \right\|$$

$$= \left\| \int_{x_0}^{x} \left(f\left(t, g(t)\right) - f\left(t, g_1(t)\right) \right) dt \right\|$$

$$\leq \int_{x_0}^{x} \left\| f\left(t, g(t)\right) - f\left(t, g_1(t)\right) \right\| dt$$

$$\leq q \int_{x_0}^{x} \|g(t) - g_1(t)\| \, dt$$

$$= q(x - x_0) \|g - g_1\|$$

$$\leq pq \|g - g_1\|$$

$$\|Tg - Tg_1\| \leq k \|g - g_1\| ,$$

where 0 < k = pq < 1.
Therefore, T is a self-contraction mapping of X. Consequently, according to the Banach contraction the theorem, T has one unique fixed point g ∗ ∈ X. The one and only solution to equation is this distinct fixed point g ∗. This extension of Banach contraction theorem was established by Boyd and Wong in 1969.

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Assumption 8 This full metric space is denoted by (X, d) and ψ : [0, ∞) → [0, ∞) have a right-to-left upper semicontinuity such that 0 ≤ ψ(t) < t for all t > 0. If T : X → X satisfies
d (T(x), T(y)) ≤ ψ (d(x, y)) ∀ x, y ∈ X,
A single fixed point x¯ ∈ X is present, and for any x ∈ X, {T n (x)} converges to x.
Proof. For any fixed x ∈ X, let xn = T n (x), n = 1, 2, . . . and an = d(xn, xn+1) = d T^n (x), T^{n+1}(x) . The convergence of is shown by us. For any positive integers n, we may presumptively say that a > 0.
So, keeping in mind that n > 1,

$$\begin{aligned} a_n &= d\left(T^n(x), T^{n+1}(x)\right) \\ &= d\left(T(x_{n-1}), T(x_n)\right) \\ &\le \psi\left(d(x_{n-1}, x_n)\right) \\ &= \psi(a_{n-1}) \\ &< a_{n-1}. \end{aligned}$$

Because it is monotonically decreasing and confined below, the sequence {an} is convergent.
Let

$$\lim_{n\to\infty} a_n = a.$$

We show that a = 0. If a > 0, then an+1 ≤ ψ(an). By the upper semicontinuity from the right of the function ψ, we obtain a ≤ ψ(a) which is a contradiction with the property of ψ. Thus a = 0 and $a_n \to 0$ as n → ∞. It is said that {xn} is a Cauchy sequence. On the other hand, let's pretend that the sequence {xn} is different from Cauchy. If ε is greater than 0, then for every k in N, there is mk > nk ≥ k.
d (xmk , xnk) ≥ ε
Additionally, let's pretend that are true for every k if and only if mk is the smallest positive integer bigger than nk.
Let
ak = d(xmk , xnk). Since

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} a_n = 0,$$

Some value of k0 is such that d(xk, xk+1) ≤ ε for all k ≥ k0. For such k, we have

$$\begin{aligned} \varepsilon &\le d(x_{m_k}, x_{n_k}) \\ &\le d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \\ &\le d(x_{m_k}, x_{m_k-1}) + \varepsilon \\ &\le d(x_k, x_{k-1}) + \varepsilon. \end{aligned}$$

This proves

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \lim_{k\to\infty} a_k = \varepsilon.$$

In contrast, we are in a

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$$
\begin{aligned}
d(x_{m_k}, x_{n_k}) &\le d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) \\
&\le a_{m_k} + \psi\left(d(x_{m_k}, x_{n_k})\right) + a_{n_k} \qquad where \quad a_{m_k} = d(x_{m_k}, x_{m_k+1}) \\
&\le 2a_k + \psi\left(d(x_{m_k}, x_{n_k})\right).
\end{aligned}
$$

As k → ∞, we obtain

$$
\begin{aligned}
\varepsilon &= \lim_{k\to\infty} d(x_{m_k}, x_{n_k}) \\
&\le \lim_{k\to\infty} \left(2a_k + \psi\left(d(x_{m_k}, x_{n_k})\right)\right) \\
&= \psi(\varepsilon).
\end{aligned}
$$

Thus ε ≤ ψ(ε) something contradicts itself. Hence As {xn} is a Cauchy sequence, {T n (x)} is as well. Given the completeness of X and the Cauchy sequence {T n (x)},

$$\lim_{n\to\infty} T^n(x) = \overline{x} \in X.$$

With T being continuous, we can write T(x) as x. As a result of x is unique.

Remark 9. Theorem 8 may not hold if we substitute the condition ψ(t) < t with the condition ψ(t0) < t0 for at least one value of t0. In such a situation, T can have several fixed points or none at all.

V. CONCLUSION

Fixed point theorems in metric spaces endowed with binary relations have emerged as powerful tools for analyzing nonlinear problems, particularly in the study of differential equations. By incorporating binary relations into the underlying metric structure, these theorems generalize classical fixed point results and allow for greater flexibility in handling ordered, comparable, or relationally constrained systems. This framework is especially effective in situations where standard contraction conditions or global order structures are not satisfied.

The application of such fixed point results to differential equations provides robust criteria for the existence, uniqueness, and stability of solutions. Through appropriate operator formulations, many classes of ordinary and partial differential equations can be transformed into fixed point problems in relational metric spaces. This approach not only unifies various existence results but also simplifies proofs by relying on relational contractive conditions rather than stronger continuity or compactness assumptions.

Overall, fixed point theory in metric spaces with binary relations significantly broadens the scope of solvable differential equations and offers a versatile mathematical framework for future research. Its continued development is expected to yield further applications in nonlinear analysis, integral equations, dynamical systems, and mathematical models arising in applied sciences.

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